Control points of a Bézier curve approximating a small circular arc

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Bernstein polynomial

$$B_{i,n}(t)=\binom{n}{i}t^i(1-t)^{(n-i)}$$

Bézier-Bernstein curve

$$C(t)=\sum_{i=0}^{n}P_iB_{i,n}(t)$$

Consider the expansion of a four point Bézier

$$\begin{aligned}&\mathrm{C}(t)=(1-t)^3P_0+3t(1-t)^2P_1+3t^2(1-t)P_2+t^3P_3\\&\mathrm{C}(t)=P_0+3(P_1-P_0)t+3(P_2-2P_1+P_0)t^2+(P_3-3P_2+3P_1-P_0)t^3\end{aligned}\qquad(1)$$

$$\begin{aligned}&\mathrm{C}(0)=P_0\\&\mathrm{C}(1)=P_3\end{aligned}$$

and its derivatives at zero and one

$$\begin{aligned}&\mathrm{C}'(t)=-3(1-t)^2P_0+3(1-t)(1-3t)P_1+3t(2-3t)P_2+3t^2P_3\\&\mathrm{C}'(t)=3(P_1-P_0)+6(P_2-2P_1+P_0)t+3(P_3-3P_2+3P_1-P_0)t^2\end{aligned}\qquad(2)$$

$$\begin{aligned}&\mathrm{C}'(0)=3(P_1-P_0)\\&\mathrm{C}'(1)=3(P_3-P_2)\end{aligned}$$

This indicates segment P_0P_1 is tanget to C at P_0, as is P_3P_2 at P_3.

Consider a unit arc A of sweep θ, < 90°, bisected by the x-axis. Let $\phi=\frac{\theta}{2}$

Force the endpoints of C to the endpoints of A. Note the symmetry due to bisection.

$$\begin{aligned}&x_0=\cos(\phi)\qquad x_3=\cos(\phi)\qquad x_3=x_0\\&y_0=\sin(\phi)\qquad y_3=-\sin(\phi)\qquad y_3=-y_0\end{aligned}\qquad(3)$$

Force the midtime of C to the midpoint of A.

$$\mathrm{P}(\frac{1}{2})=(\frac{1}{2})^3P_0+3(\frac{1}{2})(\frac{1}{2})^2P_1+3(\frac{1}{2})^2(\frac{1}{2})P_2+P_3=(1,0)\qquad(4)$$

Substitute x_3 with x_0 and y_3 with $-y_0$

$$\begin{aligned}&\frac{1}{8}x_0+\frac{3}{8}x_1+\frac{3}{8}x_2+\frac{1}{8}x_3=1\qquad\frac{1}{8}y_0+\frac{3}{8}y_1+\frac{3}{8}y_2+\frac{1}{8}y_3=0\\&2x_0+3x_1+3x_2=8\qquad 3y_1+3y_2=0\\&3x_2=8-2x_0-3x_1\qquad y_2=-y_1\end{aligned}\qquad(5)$$

Force slopes of end point tangents of C to coincide with those of A .

$$m_0 = \frac{y_0}{x_0} \quad m_0^t = \frac{-x_0}{y_0} = \frac{(y_0 - y_1)}{(x_0 - x_1)} \quad -x_0^2 + x_0 x_1 = y_0^2 - y_0 y_1 \quad x_0 x_1 = x_0^2 + y_0^2 - y_0 y_1 = 1 - y_0 y_1 \tag{6}$$

$$m_3 = \frac{y_3}{x_3} \quad m_3^t = \frac{-x_3}{y_3} = \frac{(y_3 - y_2)}{(x_3 - x_2)} \quad -x_3^2 + x_2 x_3 = y_3^2 - y_2 y_3 \quad x_2 x_3 = x_3^2 + y_3^2 - y_2 y_3 = 1 - y_2 y_3 \tag{7}$$

In (7) substitute $x_3 \text{ with } x_0, y_3 \text{ with} - y_0, \text{ and } y_2 \text{ with} - y_1$ and contrast with (6)

$x_0 x_2 = 1 - y_0 y_1 = x_0 x_1,$ thus $x_1 = x_2$ and substituting in (5) gives $x_1 = \frac{4 - x_0}{3}$

In (6) substitute x_1 and multiply through by 3

$$x_0(4 - x_0) = 3 - 3 y_0 y_1$$

$$y_1 = \frac{3 - x_0(4 - x_0)}{3 y_0} = \frac{(1 - x_0)(3 - x_0)}{3 y_0}$$

In final:

$$\begin{array}{lll} P_0 & x_0 = \cos(\frac{\theta}{2}) & y_0 = \sin(\frac{\theta}{2}) \\ P_1 & x_1 = \frac{4 - x_0}{3} & y_1 = \frac{(1 - x_0)(3 - x_0)}{3 y_0} \\ P_2 & x_2 = x_1 & y_2 = -y_1 \\ P_3 & x_3 = x_1 & y_3 = -y_0 \end{array}$$

Use rotation, scaling and translation transformations on P to make the Bezier curve approximate a circular arc of sweep θ of an arbitrarily positioned and sized circle.